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Kac–van Moerbeke equations associated with two-dimensional SU($n+1$) periodic Toda lattices

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Abstract. Various results known for one-dimensional periodic Toda lattice equations are generalised to two dimensions. In particular, a generalisation of the Kac–van Moerbeke equations is derived from a set of first-order differential equations, of which the zero gauge field strength is an integrability condition. The generalised equations are shown to be the unification of two different periodic Toda lattice equations and they naturally produce the Bäcklund transformation. The two lattice equations are simultaneously derived from a pair of $2(n+1)\times 2(n+1)$ potentials satisfying the zero field strength condition.

1. Introduction

The Toda lattice equations governed by SU($n+1$) have much richer properties than those governed by other Lie groups. For example, if the $n\times n$ Cartan matrix (K_{ij}) in the SU($n+1$) Toda lattice equation is generalised so that the non-zero entries are

$$K_{ii}=2 \qquad K_{ii+1}=K_{ii-1}=-1$$

where the indices are now defined modulo n, then the SU(n) periodic Toda lattice equation is obtained.

Furthermore, there is a set of first-order one-dimensional equations‡ considered by Kac and van Moerbeke (1975) which connects two SU($n+1$)-type Toda lattice equations. In other words, the Kac–van Moerbeke equations (hereafter referred to as the KvM equations) are simultaneously equivalent to two SU($n+1$) Toda lattice equations (Toda and Wadati 1975), one of which can be regarded as a Bäcklund transformation of the other (Wadati and Toda 1975).

In this paper, we show that a similar phenomenon occurs in two dimensions; that is, we find a generalisation of the KvM equations and describe the circumstances under which they reduce to a pair of two-dimensional Toda lattice equations. In the course of this investigation we rederive the Bäcklund transformations given by Fordy and Gibbons (1980) and Leznov *et al* (1980).

Before entering into our main discussion we shall summarise some points of the one-dimensional KvM theory to facilitate later comparison with our two-dimensional case.

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‡ This set of equations has been considered in the different context of plasma physics by Zakharov *et al* (1974).

The one-dimensional KvM equation is of the form

$$\partial_t S_i = e^{-S_{i+1}} - e^{-S_{i-1}}. \tag{1.1}$$

If two kinds, Q_k and Q_k^{B}, of Toda's displacement variable are introduced by

$$S_{2k} = Q_k^{\mathrm{B}} - Q_k \qquad S_{2k+1} = Q_{k+1} - Q_k^{\mathrm{B}} \tag{1.2}$$

so that

$$S_{2k} + S_{2k+1} = Q_{k+1} - Q_k \qquad S_{2k+1} + S_{2(k+1)} = Q_{k+1}^{\mathrm{B}} - Q_k^{\mathrm{B}} \tag{1.3}$$

then equation (1.1) may be decomposed as

$$\begin{aligned} \partial_t Q_k &= \exp(-S_{2k-1}) + \exp(-S_{2k}) \\ \partial_t Q_k^{\mathrm{B}} &= \exp(-S_{2k}) + \exp(-S_{2k+1}). \end{aligned} \tag{1.4}$$

By differentiating (1.4) with respect to t and using (1.3), Q_k and Q_k^{B} may be shown to satisfy individually the following Toda lattice equations

$$\begin{aligned} \partial_t^2 Q_k &= \exp[-(Q_k - Q_{k-1})] - \exp[-(Q_{k+1} - Q_k)] \\ \partial_t^2 Q_k^{\mathrm{B}} &= \exp[-(Q_k^{\mathrm{B}} - Q_{k-1}^{\mathrm{B}})] - \exp[-(Q_{k+1}^{\mathrm{B}} - Q_k^{\mathrm{B}})]. \end{aligned} \tag{1.5}$$

By substituting (1.2) and (1.4), it is easy to see that (1.4) represents a Bäcklund transformation for the equations (1.5).

A further attraction of the theory is that the KvM equation (1.1) may be derived from the Lax pair

$$\partial_t L = [B, L] \tag{1.6}$$

where the only non-zero entries in the matrices L and B are respectively

$$L_{ii+1} = L_{i+1i} = a_i \qquad B_{ii+2} = -B_{i+2i} = a_i a_{i+1}$$

with $a_i = \frac{1}{2}\exp(-\frac{1}{2}S_i)$ (Kac and van Moerbeke 1975, Moser 1975). Moreover the two Toda lattice equations (1.5) remarkably may be obtained from the even and odd diagonal entries in the associated Lax pair (Moser 1975)

$$\partial_t L^2 = [B, L^2]. \tag{1.7}$$

If we restrict the index k in (1.5) so that it is defined modulo $(n+1)$, then Q_k and Q_k^{B} each satisfy a one dimensional SU$(n+1)$ periodic Toda lattice equation. Our aim in this paper is to determine the two-dimensional generalisations of (1.4) when k is periodic and hence generalise all the above aspects of the KvM theory.

The paper is arranged as follows. In § 2 the two-dimensional SU(n) periodic Toda lattice equation is derived by extending the potentials B_u and $B_{\bar{u}}$ defined in our previous paper, (Farwell and Minami 1982) which we shall refer to as I. The terminology 'main equation' and 'subsidiary equations' of I will be used again here. In § 3, by using a special solution of the main equation, the set of subsidiary equations turns out to be a set of KvM equations. As a consequence, two sets of variables, both of which satisfy the two-dimensional SU$(n+1)$ periodic Toda lattice equations, are found. Section 4 includes a discussion of a Bäcklund transformation, where the Lie-type transformation is included automatically. In § 5 we propose a generalisation of the Lax pair (1.7) by specifying a $2(n+1)\times 2(n+1)$ representation of the potentials, which simultaneously gives the two Toda lattice equations. However, a detailed proof of the derivation is reserved for the appendix. The final section contains a summary of the various aspects of the theory of two-dimensional periodic Toda lattices that we have considered and also comments on this theory as a whole.

2. Preliminaries. Pair of potentials which give rise to periodic SU($n+1$) Toda lattices

In I it was the potentials B_u and $B_{\bar{u}}$ which gave rise to the non-periodic Toda lattice equations via the zero field strength condition. By applying the method of Bogoyavlensky (1976), these potentials can easily be converted to those which in a similar way give the periodic SU($n+1$) Toda lattice equations.

The set π^+ of positive simple roots of SU($n+1$) is extended to include the negative maximal root, $-\mu$. We denote the extended set by

$$\bar{\pi} = \{\pi^+, -\mu\}.$$

With respect to the ordering of the roots described in appendix 1 in I, μ corresponds to the $(n+1)$th element of $\bar{\pi}$ and so the Chevalley basis for a_n is extended to include

$$H_{n+1} = e_{n+1\,n+1} - e_{11} \qquad E_{n+1} = -e_{n+1\,1} \qquad E_{-(n+1)} = -e_{1\,n+1}.$$

By analogy with (2.19) of I the extended potentials are given by

$$\begin{aligned} B_u &= \sum_{\alpha\in\bar{\pi}}\left[y_\alpha \exp\left(a \sum_{\beta\in\bar{\pi}} K_{\alpha\beta}\psi_\beta\right)E_{-\alpha} + a(\partial_u\psi_\alpha)H_\alpha\right] \\ B_{\bar{u}} &= -\sum_{\alpha\in\bar{\pi}}\left[\tilde{y}_\alpha \exp\left(\tilde{a} \sum_{\beta\in\bar{\pi}} K_{\alpha\beta}\tilde{\psi}_\beta\right)E_{+\alpha} + \tilde{a}(\partial_{\bar{u}}\tilde{\psi}_\alpha)H_\alpha\right]. \end{aligned} \tag{2.1}$$

In (2.1), a and $\tilde{a}$ are constant scaling factors, the implication of which will be discussed in § 4; and K is the *extended* $(N+1)\times(n+1)$ Cartan matrix for SU($n+1$). For $n>1$ K has non-zero entries as follows

$$K_{ii} = 2 \qquad K_{i+1\,i} = K_{i\,i+1} = -1 \tag{2.2a}$$

where the index i is defined modulo $(n+1)$; and for $n=1$

$$K_{11} = K_{22} = -K_{12} = -K_{21} = 2. \tag{2.2b}$$

By substituting the potentials B_u and $B_{\bar{u}}$ into the zero field strength condition, we obtain, as before, a pair of subsidiary equations and a main equation. The former may be solved for y_α and $\tilde{y}_\alpha$ and the solution substituted in the original form of B_u and $B_{\bar{u}}$ to give the analogue of (2.28) in I, namely

$$\begin{aligned} B_u &= \sum_{\alpha\in\bar{\pi}}\left[\exp\left(-\tilde{a} \sum_{\beta\in\bar{\pi}} K_{\alpha\beta}\tilde{\psi}_\beta\right)E_{-\alpha} + a(\partial_u\psi_\alpha)H_\alpha\right] \\ B_{\bar{u}} &= -\sum_{\alpha\in\bar{\pi}}\left[\exp\left(-a \sum_{\beta\in\bar{\pi}} K_{\alpha\beta}\psi_\beta\right)E_{+\alpha} + \tilde{a}(\partial_u\tilde{\psi}_\alpha)H_\alpha\right]. \end{aligned} \tag{2.3}$$

Since the extended Cartan matrix is singular, it is convenient here to use the alternative Toda lattice variables

$$\sigma_\alpha = -\sum_{\beta\in\bar{\pi}} K_{\alpha\beta}\psi_\beta \qquad \tilde{\sigma}_\alpha = -\sum_{\beta\in\bar{\pi}} K_{\alpha\beta}\tilde{\psi}_\beta. \tag{2.4}$$

Furthermore if we define

$$\rho_\alpha = a\sigma_\alpha + \tilde{a}\tilde{\sigma}_\alpha \tag{2.5}$$

then, after substituting for y_α and $\tilde{y}_\alpha$, the main equation becomes

$$\partial_u \partial_{\bar{u}} \rho_\alpha = -\sum_{\beta \in \bar{\pi}} K_{\alpha\beta} \exp \rho_\beta \tag{2.6}$$

that is, the periodic Toda lattice equation.

As we remarked in the final section of I, we can consider the set of linear equations

$$\mathrm{d}\theta = \theta\omega \tag{2.7}$$

as an alternative to the zero field strength condition. In (2.7) θ is a row of 0-forms and ω is the matrix connection 1-form.

To rewrite (2.7) in terms of components, the potentials B_u and $B_{\bar{u}}$ are used in the following matrix form:

$$B_{\bar{u}} = \begin{pmatrix} ad_1 & & & & \tilde{c}_1^{\tilde{a}} \\ \tilde{c}_2^{\tilde{a}} & ad_2 & & & \\ & \tilde{c}_3^{\tilde{a}} & \ddots & & \\ & & \ddots & \ddots & \\ & & & \tilde{c}_{n+1}^{\tilde{a}} & ad_{n+1} \end{pmatrix} \qquad B_{\bar{u}} = -\begin{pmatrix} \tilde{a}\tilde{d}_1 & c_1^{a} & & & \\ & \tilde{a}\tilde{d}_2 & c_2^{a} & & \\ & & \ddots & \ddots & \\ & & & \ddots & c_n^{a} \\ c_{n+1}^{a} & & & & \tilde{a}\tilde{d}_{n+1} \end{pmatrix} \tag{2.8}$$

where d_i, $\tilde{d}_i$, c_i and $\tilde{c}_i$ are given by

$$d_{i+1} - d_i = \partial_u \sigma_i \qquad \tilde{d}_{i+1} - \tilde{d}_i = \partial_{\bar{u}} \tilde{\sigma}_i \tag{2.9}$$

$$c_i^{a} = \tilde{y}_i \, \mathrm{e}^{\tilde{a}\tilde{\sigma}_i} \qquad \tilde{c}_i^{\tilde{a}} = y_{i-1} \, \mathrm{e}^{a\sigma_{i-1}} \tag{2.10}$$

and the non-specified entries are all zero. In (2.9), (2.10) and hereinafter, all indices i are defined modulo $(n+1)$. Since

$$\omega = B_u \, \mathrm{d}u + B_{\bar{u}} \, \mathrm{d}\bar{u}$$

equations (2.7) become

$$\partial_u \theta_i = ad_i\theta_i + \tilde{c}_{i+1}^{\tilde{a}} \theta_{i+1} \tag{2.11a}$$

$$\partial_{\bar{u}} \theta_i = -\tilde{a}\tilde{d}_i\theta_i - c_{i-1}^{a} \theta_{i-1}. \tag{2.11b}$$

After differentiating (2.11a) with respect to $\bar{u}$ and (2.11b) with respect to u and then equating coefficients of θ_{i-1}, θ_{i+1} and θ_i we obtain the following equations

$$\partial_u c_i^{a} = ac_i^{a}(d_{i+1} - d_i) \qquad \partial_{\bar{u}} \tilde{c}_i^{\tilde{a}} = \tilde{a}\tilde{c}_i^{\tilde{a}}(\tilde{d}_i - \tilde{d}_{i-1}) \tag{2.12}$$

and

$$\tilde{a}\partial_u \tilde{d}_i + a\partial_{\bar{u}} d_i = c_i^{a}\tilde{c}_{i+1}^{\tilde{a}} - c_{i-1}^{a}\tilde{c}_i^{\tilde{a}}. \tag{2.13}$$

We subtract from (2.13) the similar equation with $i \to (i-1)$ to obtain

$$\tilde{a}\partial_u(\tilde{d}_i - \tilde{d}_{i-1}) + a\partial_{\bar{u}}(d_i - d_{i-1}) = c_i^{a}\tilde{c}_{i+1}^{\tilde{a}} - 2c_{i-1}^{a}\tilde{c}_i^{\tilde{a}} + c_{i-2}^{a}\tilde{c}_{i-1}^{\tilde{a}}. \tag{2.14}$$

Also the pair of equations (2.12) may be manipulated to give

$$\partial_u c_i = c_i(d_{i+1} - d_i) \qquad \partial_{\bar{u}} \tilde{c}_i = \tilde{c}_i(\tilde{d}_i - \tilde{d}_{i-1}) \tag{2.15}$$

and then substitution of the expressions (2.9) into (2.15) enables us to solve for c_i and $\tilde{c}_i$. Specifically

$$c_i = \varepsilon_i\, e^{\sigma_i} \qquad \tilde{c}_i = \tilde{\varepsilon}_i\, e^{\tilde{\sigma}_{i-1}}. \tag{2.16}$$

To conform with (2.3), we choose $\varepsilon_i = \tilde{\varepsilon}_i = 1$ for all i. By using these solutions and again the expressions (2.9), (2.14) becomes the periodic Toda lattice equation (2.6). So to use our previous terminology, (2.12) are the subsidiary equations and (2.14) is the main equation.

3. Derivation of a generalised KvM equation

In § 2 we have shown that it is possible to derive the periodic SU$(n+1)$ Toda lattice equation by using the set of linear equations (2.7) or equivalently by using the zero-curvature condition. In both cases, we solve the pair of subsidiary equations and substitute the solution in the main equation to give the Toda lattice equation. In this section we are interested in the outcome of, conversely, solving the main equation and substituting the solution in the subsidiary equations. Do the resultant subsidiary equations replace the Toda lattice equation?

It is easily checked that a special solution of the main equation (2.14) is given by

$$\begin{aligned} d_{i+1} - d_i &= \tilde{b}^{-1}(\tilde{c}^{\tilde{a}}_{i+1} - \tilde{c}^{\tilde{a}}_i) \\ \tilde{d}_{i-1} - \tilde{d}_i &= b^{-1}(c^a_{i-1} - c^a_i) \end{aligned} \tag{3.1}$$

provided that the constants a, $\tilde{a}$, b and $\tilde{b}$ satisfy

$$a\tilde{a}(b\tilde{b})^{-1} = 1. \tag{3.2}$$

Now substituting the solution (3.1) into the subsidiary equations (2.12) gives

$$\partial_u c^a_i = a\tilde{b}^{-1} c^a_i (\tilde{c}^{\tilde{a}}_{i+1} - \tilde{c}^{\tilde{a}}_i) \tag{3.3a}$$

$$\partial_{\bar{u}} \tilde{c}^{\tilde{a}}_i = \tilde{a} b^{-1} \tilde{c}^{\tilde{a}}_i (c^a_i - c^a_{i-1}). \tag{3.3b}$$

Differentiation of (3.3a) with respect to $\bar{u}$ and (3.3b) with respect to u and use of (3.1) and (3.2) produces the respective equations

$$\partial_{\bar{u}}\partial_u(\ln c^a_i) = \tilde{c}^{\tilde{a}}_{i+1}(c^a_{i+1} - c^a_i) - \tilde{c}^{\tilde{a}}_i(c^a_i - c^a_{i-1}) \tag{3.4a}$$

$$\partial_u\partial_{\bar{u}}(\ln \tilde{c}^{\tilde{a}}_i) = c^a_i(\tilde{c}^{\tilde{a}}_{i+1} - \tilde{c}^{\tilde{a}}_i) - c^a_{i-1}(\tilde{c}^{\tilde{a}}_i - \tilde{c}^{\tilde{a}}_{i-1}). \tag{3.4b}$$

By adding (3.4a) and (3.4b), we obtain

$$\begin{aligned} \partial_u\partial_{\bar{u}}[\ln(c^a_i\tilde{c}^{\tilde{a}}_i)] &= \tilde{c}^{\tilde{a}}_{i+1}c^a_{i+1} - 2\tilde{c}^{\tilde{a}}_i c^a_i + \tilde{c}^{\tilde{a}}_{i-1}c^a_{i-1} \\ &= -K_{ij}\exp[\ln(c^a_j\tilde{c}^{\tilde{a}}_j)]. \end{aligned} \tag{3.5}$$

It is a direct consequence of (3.5) that the variable

$$\rho_i = \ln(c^a_i\tilde{c}^{\tilde{a}}_i) \tag{3.6}$$

satisfies the periodic Toda lattice equation.

However, a remarkable feature of the subsidiary equations (3.4) is seen by considering, rather than (3.5), the combination

$$\begin{aligned} \partial_u\partial_{\bar{u}}[\ln(c^a_i) + \ln(\tilde{c}^{\tilde{a}}_{i+1})] &= \tilde{c}^{\tilde{a}}_i c^a_{i-1} - 2\tilde{c}^{\tilde{a}}_{i+1}c^a_i + \tilde{c}^{\tilde{a}}_{i+2}c^a_{i+1} \\ &= -K_{ij}\exp[\ln(\tilde{c}^{\tilde{a}}_{j+1}c^a_j)]. \end{aligned} \tag{3.7}$$

Hence the variable

$$\rho_i^{\mathrm{B}} = \ln(c_i^a \tilde{c}_{i+1}^{\tilde{a}}) \tag{3.8}$$

also satisfies the periodic Toda lattice equation. So, by considering different combinations of variables, equations (3.3) simultaneously produce two periodic Toda lattice equations. This situation parallels the one-dimensional example described in § 1, in which Toda and Wadati (1975) were able to produce from the KvM equations two different variables satisfying the Toda lattice equation. Hence we consider the equations (3.3) as a generalisation to two dimensions of the KvM equations (1.1).

We note that if c_i and $\tilde{c}_i$ are given by (2.16) then, using (3.2), the subsidiary equations (3.3) become

$$\partial_u \sigma_i = \tilde{b}^{-1}(\mathrm{e}^{\tilde{a}\tilde{\sigma}_i} - \mathrm{e}^{\tilde{a}\tilde{\sigma}_{i-1}}) \qquad \partial_{\tilde{u}} \tilde{\sigma}_i = b^{-1}(\mathrm{e}^{a\sigma_{i+1}} - \mathrm{e}^{a\sigma_i}) \tag{3.9}$$

and the two Toda lattice variables ρ_i and ρ_i^{B} are

$$\rho_i = a\sigma_i + \tilde{a}\tilde{\sigma}_{i-1} \tag{3.10a}$$

$$\rho_i^{\mathrm{B}} = a\sigma_i + \tilde{a}\tilde{\sigma}_i. \tag{3.10b}$$

To summarise, we have shown that by substituting a special solution of the main equation into the subsidiary equations we obtain a generalisation of the KvM equations. These are equivalent to two Toda lattice equations. It should be remarked that the equations (3.9) and (3.3) respectively can be transcribed to those given by Fordy and Gibbons (1980) and Leznov *et al* (1980) in a similar context. A trivial difference is that they use one variable with odd and even suffixes where we use two variables, one with a tilde and the other without, which naturally arise from our previous work.

4. Bäcklund transformations of SU($n+1$) periodic Toda lattices

To understand the constants (a, $\tilde{a}$, b and $\tilde{b}$) and the two Toda lattice variables (ρ_i and ρ_i^{B}) of § 3, we shall initially restrict our analysis to the periodic SU(2) Toda lattice, that is, when $n = 1$.

In this case all indices will be defined modulo 2 and the extended Cartan matrix is given by (2.2b). For this specific example, from (2.4),

$$\sigma_1 = 2(\psi_2 - \psi_1) \qquad \sigma_2 = 2(\psi_1 - \psi_2)$$

$$\tilde{\sigma}_1 = 2(\tilde{\psi}_2 - \tilde{\psi}_1) \qquad \tilde{\sigma}_2 = 2(\tilde{\psi}_1 - \tilde{\psi}_2)$$

so that

$$\sigma_2 = -\sigma_1 \equiv \sigma \qquad \tilde{\sigma}_2 = -\tilde{\sigma}_1 \equiv \tilde{\sigma}. \tag{4.1}$$

By using the identities (4.1) in the equations (3.9) for SU(2) gives

$$\partial_u \sigma = 2\tilde{b}^{-1} \sinh \tilde{a}\tilde{\sigma} \qquad \partial_{\tilde{u}} \tilde{\sigma} = -2b^{-1} \sinh a\sigma. \tag{4.2}$$

Moreover, the two Toda lattice variables are from (3.10)

$$\rho_1 = -\rho_2 = -a\sigma + \tilde{a}\tilde{\sigma} \qquad \rho_1^{\mathrm{B}} = -\rho_2^{\mathrm{B}} = -(a\sigma + \tilde{a}\tilde{\sigma}). \tag{4.3}$$

As we have demonstrated in § 3, both of these variables satisfy the SU(2) Toda lattice equation, which may be written as

$$\partial_u \partial_{\tilde{u}} \rho = -2 \sinh 2\rho$$

the sinh–Gordon equation.

From (4.3), we can show that

$$a\sigma = -\tfrac{1}{2}(\rho_1 + \rho_1^{\mathrm{B}}) \qquad \tilde{a}\tilde{\sigma} = \tfrac{1}{2}(\rho_1 - \rho_1^{\mathrm{B}})$$

and hence after substituting these expressions in (4.2), we obtain

$$\tfrac{1}{2}\partial_u(\rho_1 + \rho_1^{\mathrm{B}}) = -2\gamma \sinh \tfrac{1}{2}(\rho_1 - \rho_1^{\mathrm{B}}) \tag{4.4a}$$

$$\tfrac{1}{2}\partial_{\tilde{u}}(\rho_1 - \rho_1^{\mathrm{B}}) = 2\gamma^{-1} \sinh \tfrac{1}{2}(\rho_1 + \rho_1^{B}). \tag{4.4b}$$

In (4.4) we have written

$$\gamma \equiv a\tilde{b}^{-1} = \tilde{a}^{-1} b \tag{4.5}$$

where the equality in (4.5) is derived from the condition (3.2). The pair of coupled first-order differential equations (4.4) is the well known Bäcklund transformation of the sinh–Gordon equation. Hence, at least for the $n = 1$ case, the two Toda lattice variables are related by a Bäcklund transformation. Furthermore, the identities (4.5) suggest that the scaling constants in the combinations $a\tilde{b}^{-1}$ and $\tilde{a}^{-1}b$ are related to the Lie transformation.

We now consider the subsidiary equations (3.9) for arbitrary n: it is more convenient to use Toda's original displacement variables, q_i and q_i^{B}, to rewrite ρ_i and ρ_i^{B} respectively by the formulae

$$\rho_i = q_{i-1} - q_i \qquad \rho_i^{\mathrm{B}} = q_{i-1}^{\mathrm{B}} - q_i^{\mathrm{B}}. \tag{4.6}$$

By using the relations (3.10), this is equivalent to defining

$$\sigma_i = a^{-1}(q_{i-1}^{\mathrm{B}} - q_i) \qquad \tilde{\sigma}_i = \tilde{a}^{-1}(q_i - q_i^{\mathrm{B}}) \tag{4.7}$$

which is a natural generalisation of the one-dimensional definitions (1.2).

By substituting (4.7) into (3.9), we directly obtain

$$\partial_u(q_i^{\mathrm{B}} - q_{i+1}) = -\gamma[\exp(q_i - q_i^{\mathrm{B}}) - \exp(q_{i+1} - q_{i+1}^{\mathrm{B}})] \tag{4.8a}$$

$$\partial_{\tilde{u}}(q_i - q_i^{\mathrm{B}}) = \gamma^{-1}[\exp(q_i^{\mathrm{B}} - q_{i+1}) - \exp(q_{i-1}^{\mathrm{B}} - q_i)]. \tag{4.8b}$$

This is the Bäcklund transformation for the SU($n+1$) Toda lattice which was obtained by Fordy and Gibbons (1980) and to which Leznov *et al* (1980) alluded.

We have thus shown that starting from the set of linear equations (2.7), we can naturally obtain the Bäcklund transformation for the two-dimensional periodic Toda lattice from the generalised KvM equations (3.9). This exactly mirrors the one-dimensional case described by Toda and Wadati (1975).

5. Simultaneous derivation of the two Toda lattices

The attraction of the KvM equations is that they are simultaneously equivalent to two Toda lattice equations. Furthermore, in the one-dimensional case, it is possible to obtain these two equations from one Lax pair, given by (1.7). Therefore it is interesting to see whether there is a similar occurrence in two dimensions, when the Lax pair is generalised to the zero field strength condition. So in this section we look for a pair of potentials, D_u and $D_{\bar{u}}$, which simultaneously generate the two Toda lattice equations, via the condition

$$\partial_{\bar{u}} D_u - \partial_u D_{\bar{u}} + [D_{\bar{u}}, D_u] = 0. \tag{5.1}$$

For neatness of notation, we shall use $a = \tilde{a} = 1$, $b = \tilde{b} = 1$ throughout this section.

We propose that the potentials are $2(n+1) \times 2(n+1)$ matrices given by

$$\begin{aligned} D_u &= \sum_{k=1}^{2(n+1)} \left\{ d_{[k/2+1]}\, e_{kk} + \tilde{c}_{[(k+1)/2]}\, e_{kk-2} \right\} \\ D_{\bar{u}} &= -\sum_{k=1}^{2(n+1)} \{ \tilde{d}_{[(k+1)/2]}\, e_{kk} + c_{[k/2+1]}\, e_{kk+2} \} \end{aligned} \tag{5.2}$$

where the index k is defined modulo $2(n+1)$ and the square brackets denote the integral part of the enclosed index.

In a similar way to (2.9), we introduce the variables σ_i and $\tilde{\sigma}_i$ by

$$\begin{aligned} d_{[k/2+2]} - d_{[k/2+1]} &= \partial_u \sigma_{[k/2+1]} \\ \tilde{d}_{[(k+1)/2]} - \tilde{d}_{[(k-1)/2]} &= \partial_{\bar{u}} \tilde{\sigma}_{[(k-1)/2]}. \end{aligned} \tag{5.3}$$

The potentials (5.2) are substituted into the condition (5.1) and the resultant equations manipulated as in § 2. The details are given in the appendix and so we only quote the results here. The subsidiary equations are solved to give

$$\tilde{c}_{[(k+1)/2]} = \exp \tilde{\sigma}_{[(k-1)/2]} \qquad c_{[k/2+1]} = \exp \sigma_{[k/2+1]} \tag{5.4}$$

and then the main equation has the form

$$\begin{aligned} \partial_u \partial_{\bar{u}} (\sigma_{[k/2+1]} + \tilde{\sigma}_{[(k+1)/2]}) &= \exp(\tilde{\sigma}_{[(k-1)/2]} + \sigma_{[k/2]}) - 2 \exp(\tilde{\sigma}_{[(k+1)/2]} + \sigma_{[k/2+1]}) \\ &\quad + \exp(\tilde{\sigma}_{[(k+3)/2]} + \sigma_{[k/2+2]}) \end{aligned} \tag{5.5}$$

If in (5.5) we specify k to be odd, that is $k = 2i - 1$, then the main equation becomes

$$\partial_u \partial_{\bar{u}} (\sigma_i + \tilde{\sigma}_i) = \exp(\sigma_{i-1} + \tilde{\sigma}_{i-1}) - 2 \exp(\sigma_i + \tilde{\sigma}_i) + \exp(\sigma_{i+1} + \tilde{\sigma}_{i+1}). \tag{5.6}$$

However when k is even, that is $k = 2i$, then (5.5) becomes

$$\partial_u \partial_{\bar{u}} (\sigma_{i+1} + \tilde{\sigma}_i) = \exp(\tilde{\sigma}_{i-1} + \sigma_i) - 2 \exp(\tilde{\sigma}_i + \sigma_{i+1}) + \exp(\tilde{\sigma}_{i+1} + \sigma_{i+2}). \tag{5.7}$$

We note that for both k odd and k even. the subsidiary equations (A3), are the same and in fact are (2.12). The equations (5.6) and (5.7) are the required pair of Toda lattice equations, (3.7) and (3.5) respectively, with c_i and $\tilde{c}_i$ given by (2.16). Hence, the condition (5.1) and D_u and $D_{\bar{u}}$ given by (5.2) simultaneously give the two Toda lattices.

Now to determine a relationship between the paie of potentials (5.2) and the original potentials B_u and $B_{\bar{u}}$, given by (2.8), we shall transform D_u and $D_{\bar{u}}$ according to

$$D'_u = E D_u E^{-1} \qquad D'_{\bar{u}} = E D_{\bar{u}} E^{-1}. \tag{5.8}$$

E is the $2(n+1) \times 2(n+1)$ matrix with non-zero entries e_{kk+1} and periodic indices. The effect of the transformation (5.8) is to shift periodically the entries of the matrix one place to the left along a diagonal. This is easily seen if we write D_u and $D_{\bar{u}}$ explicitly in matrix form as

$$D_u=\begin{pmatrix} d_1 & & & & & \tilde{c}_1 & 0 \\ 0 & d_2 & & & & & \tilde{c}_1 \\ \tilde{c}_2 & 0 & d_2 & & & & \\ & \tilde{c}_2 & 0 & d_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \tilde{c}_n & 0 & d_n & \\ & & & & \tilde{c}_n & 0 & d_1 \end{pmatrix} \qquad D_{\bar{u}}=-\begin{pmatrix} \tilde{d}_1 & 0 & c_1 & & & & \\ & \tilde{d}_1 & 0 & c_2 & & & \\ & & \tilde{d}_2 & 0 & c_2 & & \\ & & & \tilde{d}_2 & 0 & c_3 & \\ & & & & \ddots & \ddots & c_n \\ c_n & & & & & \ddots & 0 \\ 0 & c_1 & & & & & \tilde{d}_n \end{pmatrix}. \tag{5.9}$$

Then

$$D'_u=\begin{pmatrix} d_2 & & & & & \tilde{c}_1 & 0 \\ 0 & d_2 & & & & & \tilde{c}_1 \\ \tilde{c}_2 & 0 & d_3 & & & & \\ & \tilde{c}_2 & 0 & d_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \tilde{c}_n & 0 & d_n & \\ & & & & \tilde{c}_n & 0 & d_1 \end{pmatrix} \qquad D'_{\bar{u}}=-\begin{pmatrix} \tilde{d}_1 & 0 & c_2 & & & & \\ & \tilde{d}_2 & 0 & c_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & c_n \\ & & & & \tilde{d}_n & 0 & c_n \\ c_1 & & & & & \tilde{d}_n & 0 \\ 0 & c_1 & & & & & \tilde{d}_1 \end{pmatrix}. \tag{5.10}$$

Since the primed potentials (5.8) also satisfy the condition (5.1), we consider the difference

$$\partial_{\bar{u}}(D_u-D'_u)-\partial_u(D_{\bar{u}}-D'_{\bar{u}})=[D_u, D_{\bar{u}}]-[D'_u, D'_{\bar{u}}]. \tag{5.11}$$

The details of the calculation are given in the appendix, but we note that

$$\begin{aligned} (D_u-D'_u)&=\sum_{i=1}^{(n+1)}[(d_i-d_{i+1})e_{2i-12i-1}+(\tilde{c}_i-\tilde{c}_{i+1})e_{2i2i-2}] \\ (D_{\bar{u}}-D'_{\bar{u}})&=\sum_{i=1}^{(n+1)}[(\tilde{d}_{i+1}-\tilde{d}_i)e_{2i2i}+(c_{i+1}-c_i)e_{2i-12i+1}]. \end{aligned} \tag{5.12}$$

Explicit calculation of (5.11) gives the two pairs of equations (A6) and (A7) from even and odd matrix entries respectively. However, if in (A6) and (A7) we make use of the solutions (3.1) which previously converted the subsidiary equations into the generalised KvM equations, then we obtain just one pair of equations (A8). From the equations (A8) we can infer that

$$\partial_u c_i=c_i(\tilde{c}_{i+1}-\tilde{c}_i) \qquad \partial_{\bar{u}}\tilde{c}_i=\tilde{c}_i(c_i-c_{i-1}) \tag{5.13}$$

which are the generalised KvM equations.

6. Concluding remarks

We now summarise the contents of this paper. Firstly, we have shown *how* the generalised KvM equations in two dimensions may be derived by using a pair of potentials subjected to the condition (2.7). An important ingredient of this derivation is the special solution (3.1) of the main equation, since it converts the subsidiary equations into the KvM equations. In addition we have shown how to obtain from the KvM equations two different variables, both of which satisfy the SU$(n+1)$ periodic Toda lattice equations. In contrast to the one-dimensional case, we did not need to use even and odd suffixes of one variable, as two different variables naturally arise from the potentials B_u and $B_{\bar{u}}$.

Secondly, by using an arrangement of variables similar to that of the one-dimensional case, we have rederived the Bäcklund transformations of Fordy and Gibbons (1980) and Leznov *et al* (1980) for the SU$(n+1)$ periodic Toda lattice equations. As expected, when $n=1$, these became the Bäcklund transformations for the sinh–Gordon equation. In addition, the constant scaling factors, which we included in our original definition (2.8) of the potentials, are shown to be related to the Lie transformations. Furthermore, we expect that these factors are connected to the conservation laws of the periodic system.

Finally, we have defined a pair of larger matrix potentials, D_u and $D_{\bar{u}}$, which, via the zero field strength condition, simultaneously produce the two different Toda lattice equations described above.

However, we should remark that we have not included any attempt to parametrise the solutions of the periodic equations. In contrast to the one-dimensional case, it is not easy to obtain even a one-soliton solution by using the Bäcklund transformations applied to the trivial solution. This is partly because, in the one-dimensional case, by putting, for example

$$Q_k^{\mathrm{B}} = \text{constant},$$

we can solve the first equation of (1.4) to obtain Q_k. In the two-dimensional case, from (4.7) and (2.16),

$$\ln c_i^a = q_{i-1}^{\mathrm{B}} - q_i \qquad \ln \tilde{c}_i^{\bar{a}} = q_{i-1} - q_{i-1}^{\mathrm{B}}$$

and so the analogues of (1.4) are the subsidiary equations (4.8). Consequently, since (4.8*a*) contains a derivative with respect to a different variable from (4.8*b*), we cannot separate the q_i and q_i^{B} and insert a trivial solution. Also the method of solution used in I is not directly applicable since the potentials (2.3) are not triangular. However, it may be possible to replace B_u and $B_{\bar{u}}$ in (2.3) by infinite periodic lower and upper triangular matrices respectively and then apply the method.

It is true that the SU$(n+1)$ periodic Toda lattice equation is a generalisation of the SU(2) equation, and hence of the sinh–Gordon equation. It is therefore interesting to enquire whether the SU$(n+1)$ periodic equations share the same properties as those possessed by the sinh–Gordon equation; for example, the geometrical properties discussed by Crampin *et al* (1977)†. However, we feel that the SU(2) case is not typical, since it is endowed with particular symmetries between the variables, as shown in § 4. Consequently an analysis of the Bäcklund transformation using the Iwasawa decomposition may not be appropriate for arbitrary n.

In conclusion, we mention that we have not discussed in depth the shifting matrix E of (5.8). In the construction of the potentials B_u and $B_{\bar{u}}$ for the periodic SU$(n+1)$ Toda lattice, it is the simple roots plus the negative of the maximal root, that is, the set $\bar{\pi}$, which play an important part. Now the matrices $E, E^2, \ldots, E^n, E^{n+1} = I$ are generating elements of a cyclic subgroup of the Weyl group for SU$(n+1)$ since they cyclically permute the diagonal elements. (We could equivalently consider the matrix $\bar{E} = E^{-1}$

† In this paper, a relationship between the sinh–Gordon Bäcklund transformation and the Iwasawa decomposition for SL$(2, R)$ is exhibited.

which shifts the matrix entries one place to the right along a diagonal.) Therefore, it is an interesting problem to investigate the role of E, or equivalently $\bar{E}$, in the theory of periodic Toda lattice equations and the KvM equations.

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Appendix.
Derivation of equation (5.5)

D_u and $D_{\bar{u}}$ are given by (5.2). Then, by using

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj} \tag{A1}$$

the commutator in the condition (5.1) is

$$\begin{aligned}[D_u, D_{\bar{u}}] = {} & \tilde{c}_{[(k+3)/2]}c_{[k/2+1]}(e_{kk} - e_{k+2k+2}) \\ & - d_{[k/2+1]}c_{[k/2+1]}e_{kk+2} + d_{[k/2+2]}c_{[k/2+1]}e_{kk+2} \\ & - \tilde{c}_{[(k+1)/2]}\tilde{d}_{[(k-1)/2]}e_{kk-2} + \tilde{c}_{[(k+1)/2]}\tilde{d}_{[(k+1)/2]}e_{kk-2}.\end{aligned} \tag{A2}$$

Hence from (5.1), by comparing coefficients of (A2) and the appropriate derivatives of (5.2) we directly obtain the subsidiary equations

$$\begin{aligned}\partial_{\bar{u}}\tilde{c}_{[(k+1)/2]} &= \tilde{c}_{[(k+1)/2]}(\tilde{d}_{[(k+1)/2]} - \tilde{d}_{[(k-1)/2)]}\tilde{d}_{[\ ]}) \\ \partial_u c_{[k/2+1]} &= c_{[k/2+1]}(d_{[k/2+2]} - d_{[k/2+1]}d_{[\ ]})\end{aligned} \tag{A3}$$

and the main equation

$$\begin{aligned}\partial_{\bar{u}}(d_{[k/2+2]} - d_{[k/2+1]}) + \partial_u(\tilde{d}_{[(k+3)/2]} - \tilde{d}_{[(k+1)/2]}) \\ = -2c_{[k/2+1]}\tilde{c}_{[(k+3)/2]} + c_{[k/2]}\tilde{c}_{[(k+1)/2]} + c_{[k/2+2]}\tilde{c}_{[(k+5)/2]}.\end{aligned} \tag{A4}$$

It is now easy to see that substituting (5.3) into (A3) and solving for $\tilde{c}_{[(k+1)/2]}$ and $c_{[k/2+1]}$ gives (5.4). Then substitution of (5.4) and (5.3) into (A4) gives (5.5).

The matrix equation (5.11)

The primed potentials (5.10) may be written formally as

$$D'_u = \sum_{k=1}^{2(n+1)} \{d_{[(k+1)/2+1]}e_{kk} + \tilde{c}_{[k/2+1]}e_{kk-2}\}$$

$$D'_{\bar{u}} = -\sum_{k=1}^{2(n+1)} \{\tilde{d}_{[k/2+1]}e_{kk} + c_{[(k+3)/2]}e_{kk+2}\}$$

and so

$$[D'_u, D'_{\bar{u}}] = \tilde{c}_{[k/2+2]}c_{[(k+3)/2]}(e_{kk} - e_{k+2k+2}) + e_{kk+2}(d_{[(k+3)/2+1]}c_{[(k+3)/2]} - d_{[(k+1)/2+1]}c_{[(k+3)/2]}) + e_{k+2k}(\tilde{d}_{[k/2+2]}\tilde{c}_{[k/2+2]} - \tilde{d}_{[k/2+1]}\tilde{c}_{[k/2+2]}). \tag{A5}$$

By using the commutators (A2) and (A5) and the expressions (5.12) in the condition (5.11) we obtain:

(i) when $k = 2i$

$$\partial_u(\tilde{d}_i - \tilde{d}_{i+1}) = c_i(\tilde{c}_{i+1} - \tilde{c}_i) - c_{i+1}(\tilde{c}_{i+2} - \tilde{c}_{i+1}) \tag{A6a}$$

$$\partial_{\bar{u}}(\tilde{c}_i - \tilde{c}_{i+1}) = \tilde{c}_i(\tilde{d}_i - \tilde{d}_{i-1}) - \tilde{c}_{i+1}(\tilde{d}_{i+1} - \tilde{d}_i) \tag{A6b}$$

(ii) when $k = 2i - 1$

$$\partial_{\bar{u}}(d_i - d_{i+1}) = \tilde{c}_i(c_i - c_{i-1}) - \tilde{c}_{i+1}(c_{i+1} - c_i) \tag{A7a}$$

$$\partial_u(c_i - c_{i+1}) = c_i(d_{i+1} - d_i) - c_{i+1}(d_{i+2} - d_{i+1}). \tag{A7b}$$

Note that for $k = 2i$ the coefficient of e_{kk+2} on the right-hand side of (5.11) is zero and similarly for the coefficient of e_{k+2k} when $k = 2i - 1$, which is consistent with the left-hand side.

Now we use the expressions (3.1) for the d and $\tilde{d}$ differences in the left-hand sides of (A6*a*) and (A7*a*) and in the right-hand sides of (A6*b*) and (A7*b*) to give the repeated set of equations

$$\begin{aligned} \partial_u(c_i - c_{i+1}) &= c_i(\tilde{c}_{i+1} - \tilde{c}_i) - c_{i+1}(\tilde{c}_{i+2} - \tilde{c}_{i+1}) \\ \partial_{\bar{u}}(\tilde{c}_i - \tilde{c}_{i+1}) &= \tilde{c}_i(c_i - c_{i-1}) - \tilde{c}_{i+1}(c_{i+1} - c_i). \end{aligned} \tag{A8}$$

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